Multi-instantons in seven dimensions

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Abstract

We consider the self-dual Yang-Mills equations in seven dimensions. Modifying the t'Hooft construction of instantons in $d = 4$, we find N-instanton $7d$ solutions which depend on $8N$ effective parameters and are E_6-invariant.

1 Introduction

The pure Yang-Mills (YM) theory defined in the four-dimensional Euclidean space has a rich and interesting structure even at the classical level. The discovery of regular solutions to the YM field equations, which correspond to absolute minimum of the action (Belavin et al.) [1], has led to an intensive study of such a classical theory. One hopes that a deep understanding of the classical theory will be invaluable when one tries to quantize such a theory.

In the past few years, increased attention has been paid to gauge field equations in space-time of dimension greater than four, with a view to obtaining physically interesting theories via dimensional reduction [2]. Such equations appear in the many-dimensional theory of supergravity, in the low-energy effective theory of d-branes, and in M-theory [3]. Using solutions of the YM equations in $d > 4$ makes possible to obtain soliton solutions in these theories [4]. It is known also that the YM theory in d dimensions may

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be reduced to the Yang-Mills-Higgs (YMH) theory in $k < d$ dimensions [5]. Hence, solutions of the YMH equations in $d = 4$ may be obtained from solutions of the YM equations in $d > 4$ dimensions.

In Ref. [6], the 4d self-dual Yang-Mills equations was generalized to the higher-dimensional linear relations (CDFN equations)

$$c_{mnps}F^{ps} = \lambda F_{mn}, \tag{1}$$

where the numerical tensor c_{mnps} is completely antisymmetric and $\lambda = const$ is a non-zero eigenvalue. It is obviously that these equations lead to the full YM equation, via the Bianchi identity. Several self-dual solutions of (1) were found in [7].

The paper is organized as follows. Sections 2 and 3 contain well-known facts about the Cayley-Dickson algebras and their derivations. In Section 4 multi-instanton solutions of the G_2-invariant CDFN equations are found. In Section 6 the E_6-invariance of these solutions are proved.

2 Cayley-Dickson algebras

Let A be an algebra with an involution $x \to \bar{x}$ over a field F of characteristic $\neq 2$. Given a nonzero $\alpha \in F$ we define a multiplication on the vector space $(A, \alpha) = A \oplus A$ by

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - \alpha\bar{y}_2y_1, y_2x_1 + y_1\bar{x}_2).$$

This makes (A, α) an algebra over F. It is clear that A is isomorphically embedded into (A, α) and $\dim(A, \alpha) = 2\dim A$. Let $e = (0, 1)$. Then $e^2 = -\alpha$ end $(A, \alpha) = A \oplus Ae$. Given any $z = x + ye$ in (A, α) we suppose $\bar{z} = \bar{x} - ye$. Then the mapping $z \to \bar{z}$ is an involution in (A, α).

Starting with the base field F the Cayley-Dickson construction leads to the following sequence of alternative algebras:

1) F, the base field.

2) $\mathbb{C}(\alpha) = (F, \alpha)$, a field if $x^2 + \alpha$ is the irreducible polynomial over F; otherwise, $\mathbb{C}(\alpha) \simeq F \oplus F$.

3) $\mathbb{H}(\alpha, \beta) = (\mathbb{C}(\alpha), \beta)$, a generalized quaternion algebra. This algebra is associative but not commutative.

4) $\mathbb{O}(\alpha, \beta, \gamma) = (\mathbb{H}(\alpha, \beta), \gamma)$, a Cayley-Dickson algebra. It is easy to prove that this algebra is nonassociative.

The algebras in 1) – 4) are called composition. Any of them has the nondegenerate quadratic form (norm) $n(x) = x\bar{x}$, such that $n(xy) = n(x)n(y)$. The norm $n(x)$ defines the scalar product

$$(x, y) = \frac{1}{2}(\bar{x}y + \bar{y}x) \tag{2}$$

that is invariant with respect to all automorphisms of the composition algebra. It is known also that over the field $\mathbb{R}$ of real numbers, the above construction gives 3 split algebras (e.g., if $\alpha = \beta = \gamma = -1$) and 4 division algebras (if $\alpha = \beta = \gamma = 1$): the fields of real $\mathbb{R}$ and complex $\mathbb{C}$ numbers, the algebras of quaternions $\mathbb{H}$ and octonions $\mathbb{O}$, taken with the Euclidean norm $n(x)$. Finally note that any composition algebra is alternative, i.e. in any of them the associator

$$(x, y, z) = (xy)z - x(yz) \tag{3}$$

is skew-symmetric over x, y, z. Note also that any simple nonassociative alternative algebra is isomorphic to the Cayley-Dickson algebra $\mathbb{O}(\alpha, \beta, \gamma)$.

As any finite-dimensional algebra, the Cayley-Dickson algebra may be defined by "a multiplication table" in some fixed basis. For that we consider a real linear space A equipped with a nondenerate symmetric metric g of signature $(8, 0)$ or $(4, 4)$. Choose the basis $1, e_1, \ldots, e_7$ in A such that

$$g = \mathrm{diag}(1, \alpha, \beta, \alpha\beta, \gamma, \alpha\gamma, \beta\gamma, \alpha\beta\gamma), \tag{4}$$

where $\alpha, \beta, \gamma = \pm 1$. Define the multiplication

$$e_i e_j = -g_{ij} + c_{ij}{}^k e_k, \tag{5}$$

where the structural constants $c_{ijk} = g_{ks} c_{ij}{}^s$ are completely antisymmetric and different from 0 only if

$$c_{123} = c_{145} = c_{167} = c_{246} = c_{275} = c_{374} = c_{365} = 1.$$

The multiplication (5) transform A into a real linear algebra. It can easily be checked that A is isomorphic to $\mathbb{O}(\alpha, \beta, \gamma)$.

3 Derivations

Recall that a derivation of an algebra A is a linear transformation D of A satisfying

$$(xy)D = (xD)y + x(yD)$$

for all $x, y \in A$. The derivations of Cayley-Dickson algebra may be described in intrinsic terms. Namely, let A be a Cayley-Dickson algebra. Then for any $x, y \in A$ the mapping

$$D_{x,y} : z \longrightarrow 2[z, [x, y]] + 6(z, x, y) \tag{6}$$

is a derivation of A. Therefore we have the linear mapping $\Lambda^2 \to \mathrm{Der}A$. Since any Cayley-Dickson algebra is simple, it follows that this mapping is surjective. In addition, the following relations

$$D_{x,yz} = D_{y,xz} + D_{z,yx}, \tag{7}$$
$$[D_{x,y}, D_{z,t}] = D_{(xD_{z,t}),y} + D_{x,(yD_{z,t})} \tag{8}$$

are true. Note also that the derivations algebra $\mathrm{Der}A$ is a simple exceptional Lie algebra of type g_2.

Since the associator (3) of Cayley-Dickson algebra is skew-symmetric over its arguments, it follows that we can define the completely antisymmetric tensor c_{ijkl} by

$$(e_i, e_j, e_k) = 2c_{ijk}{}^{l} e_l. \tag{9}$$

It is easy to prove that this tensor satisfies the following identities:

$$c_{ijs} c_{kl}{}^{s} = g_{ik} g_{jl} - g_{il} g_{jk} + c_{ijkl}, \tag{10}$$
$$c_{ijps} c_{kl}{}^{ps} = 4(g_{ik} g_{jl} - g_{il} g_{jk}) + 2c_{ijkl}; \tag{11}$$

and has the nonzero components:

$$c_{4567} = c_{2367} = c_{2345} = c_{1357} = c_{1364} = c_{1265} = c_{1274} = 1.$$

Further, it follows from (2) and (9) that the tensor c_{ijkl} is invariant with respect to all automorphisms of algebra A. Noting that the group AutA is isomorphic to the Lie group of type G_2, we see that the tensor c_{ijkl} is G_2-invariant. Finally, rewriting the identity (7) in the form

$$c_i{}^{jk} D_{jk} = 0,$$

where the derivation $\frac{1}{8} D_{e_i, e_j}$ is denoted by the symbol D_{ij}, we get the following relations

$$c_{ij}{}^{kl} D_{kl} = -2D_{ij}. \tag{12}$$

Since the algebra Der$A$ is a Lie algebra of type g_2 (or g_2' in noncompact case), it follows that it may be considered as a subalgebra of the Lie algebra $so(m,n)$ of type $so(7)$ or $so(3,4)$. Hence there exists the projector c^+_{ijkl} of one onto the subspace Der$A$. Usually this projector is chosen in the form (See [7]):

$$c^+_{ijkl} = \frac{1}{6}\left(2g_{ik}g_{lj} - 2g_{il}g_{jk} - c_{ijkl}\right). \tag{13}$$

In addition, it is easily shown that the derivations

$$D_{ij} = \frac{3}{2}c^{+kl}_{ij}E_{kl}, \tag{14}$$

where E_{kl} are generators of the Lie algebra $so(m,n)$ satisfying the switching relations

$$[E_{ij}, E_{kl}] = g_{k[i}E_{j]l} - g_{l[i}E_{j]k}.$$

Besides, it follows from (11) that

$$c_{ij}{}^{ps}c^+_{klps} = -2c^+_{ijkl}. \tag{15}$$

Comparing (14) and (15), we again obtain the identity (12).

4 Solutions

Recall that the self-dual equations has been successfully tackled by the twistor techniques, and in the case of finite action solutions by the algebraic ADHM construction [9]. A generalization of the ADHM construction for the equations (1) which break $SO(4n)$ up to $Sp(1)\times Sp(n)/Z_2$ was found in [10]. However in dimensions 7 and 8 there exists an exceptional G_2-covariant (respectively $Spin(7)$-covariant) duality which is connected with the octonionic algebra. Therefore the search of generalized ADHM construction in $d = 7$ and 8 appears very attractive.

Such attempt was done in the recent paper [11]. In one the generalized ADHM construction in $d = 8$ was built with the help of the algebra $L(\mathbb{O})$ of left multiplications of octonionic algebra $\mathbb{O}$. Unfortunately, calculating the field strength in Section 5 and proving its self-duality the authors incorrect use the equality $L(L(xy)z) = L(xyz)$, where $L(xyz) = x(yz)$ and $x, y, z \in \mathbb{O}$. By associativity of the octonionic algebra it would not be done.

Nevertheless, it is easy to get multi-instanton solutions (but not a generalized ADHM construction) of CDFN equations in seven dimensions. We choose the ansatz A_m in the form:

$$A_m = \frac{\lambda^\dagger y^i}{1 + y^\dagger y} D_{mi}, \tag{16}$$

where y is a column vector with the elements $y_1, \ldots, y_N$ of Cayley-Dickson algebra such that

$$\begin{aligned} y^\dagger &= (y_1^k, \ldots, y_N^k)\bar{e}_k, & y_I^k &\in \mathbb{R}, \\ \lambda^\dagger &= (\lambda_1, \ldots, \lambda_N), & \lambda_I &\in \mathbb{R}^+, \\ y_I^k &= (b_{IJ}^k + \delta_{IJ} x^k)\lambda_J, & b_{IJ}^k &= b_{JI}^k. \end{aligned}$$

Using the identities (8)–(10), we get the field strength

$$F_{mn} = -\frac{\lambda^\dagger \{(2 + 2y^\dagger y - y^i y_i^\dagger) D_{mn} + 3c_{mn}^{+\;is} D_{sj} y^j y_i^\dagger\}\lambda}{(1 + y^\dagger y)^2},$$

where the tensor c_{ijkl}^+ is defined dy the equality (13). Now it follows from (12) and (15) that the field strength F_{mn} satisfies the CDFN equations (1) as for Euclidean as for pseudoeuclidean metric of the form (4).

This construction of multi-instanton solutions of the CDFN equations may be easy to extend in eight dimensions. It is sufficient to take the projector f_{ijkl}^+ of the algebra Lie of type $so(8)$ or $so(4,4)$ onto the subalgebra $so(7)$ or $so(3,4)$ respectively in place c_{ijkl}^+, to define the elements D'_{ij} of the form (14), and to prove an analog of the identity (15) (See [7]). Then choosing the ansatz A'_m in the form:

$$A'_m = \frac{\lambda^\dagger y^i}{1 + y^\dagger y} D'_{mi}, \tag{17}$$

where the indexes $m, i \in \{0, \ldots, 7\}$, we can obtain the following expression for the field strength:

$$F'_{mn} = -\frac{1}{3}\frac{\lambda^\dagger \{(6 + 6y^\dagger y - 3y^i y_i^\dagger) D'_{mn} + 8f_{mn}^{+\;is} D'_{sj} y^j y_i^\dagger\}\lambda}{(1 + y^\dagger y)^2}.$$

Obviously, the N-instanton solutions (16) and (17) depend on $8N$ and $9N$ effective parameters respectively, and are a generalization of the t'Hooft solution in $d = 4$ (see e.g. [12]).

5 E_6-invariance

Let A be a real Cayley-Dickson algebra with the involution $x \to \bar{x}$, and let A_3 be the algebra of all 3×3 matrix with elements of A. Consider the set

$$J = \{(x_{ij}) \in A_3 \mid (\bar{x}_{ij}) = (x_{ji})\}.$$

The set J is a commutative nonassociative algebra with the respect to the product

$$x \circ y = \frac{1}{2}(xy + yx).$$

The algebra J satisfies the identity

$$(x^2 y)x = x^2(yx)$$

and is said to be an exceptional Jordan algebra.

Denote 3×3 matrix (x_{ij}) with the unique nonzero element $x_{ij} = 1$ by the symbol ε_{ij} and choose in J the basis:

$$\begin{aligned} E_1 &= \varepsilon_{11}, & X_1(e_i) &= e_i\varepsilon_{23} + \bar{e}_i\varepsilon_{32}, \\ E_2 &= \varepsilon_{22}, & X_2(e_j) &= e_j\varepsilon_{31} + \bar{e}_j\varepsilon_{13}, \\ E_3 &= \varepsilon_{33}, & X_3(e_k) &= e_k\varepsilon_{12} + \bar{e}_k\varepsilon_{21}, \end{aligned} \tag{18}$$

where $e_0 = 1, e_1, \ldots, e_7$ is the standard basis of A. It can easily be checked that

$$E_\alpha \circ X_\beta(e_i) = \begin{cases} 0, & \text{if} \quad \alpha = \beta, \\ \frac{1}{2}X_\beta(e_i), & \text{if} \quad \alpha \neq \beta, \end{cases} \tag{19}$$

$$X_\alpha(e_i) \circ X_\beta(e_j) = \begin{cases} \delta_{ij}(E - E_\alpha), & \text{if} \quad \alpha = \beta, \\ \frac{1}{2}X_\gamma(\bar{e}_j\bar{e}_i), & \text{if} \quad \alpha \neq \beta, \end{cases} \tag{20}$$

where E is the identity 3×3 matrix, and $(\alpha\beta\gamma) = (123), (231), (312)$.

It is well known (see e.g. [8]) that the derivations algebra Der$J$ is a simple exceptional Lie algebra of the type f_4. Since there is an isomorphic enclosure of the algebra g_2 into f_4, we can consider (16) as a field that takes its values

in Der$J$. To prove the F_4-invariance of these solutions, we find the trace of the matrix

$$X_\beta = \{(X_\alpha(e_i), X_\beta(e_j), X_\alpha(e_k)) - \frac{1}{2}(X_\alpha(e_i), X_\alpha(e_j), X_\alpha(e_k))\} \circ X_\beta(e_l), \quad (21)$$

where $i, j, k \neq 0$, and we do not sum on the recurring indexes. Using (9) and (19)–(20), we prove that

$$X_\beta = \frac{1}{2} c_{ijkl}(E - E_\beta),$$

and hence

$$\mathrm{tr} X_\beta = c_{ijkl}.$$

Since a trace of matrix in J is invariant with respect to all automorphisms of J, we prove the F_4-invariance of solutions of the corresponding CDFN equations.

Moreover, it can be proved that the tensor c_{ijkl} is E_6-invariant. Indeed, the group E_6 is a group of linear transformations of the space J that preserve the norm

$$n(X) = x_{11}x_{22}x_{33} + (x_{12}x_{23})x_{31} + x_{13}(x_{32}x_{21}) \\ - x_{11}x_{23}x_{32} - x_{22}x_{31}x_{13} - x_{33}x_{12}x_{21},$$

where $X = (x_{ij}) \in J$. Choose an element X in the form

$$X = X_1 + X_2 - X_3 + E_1 + E_2,$$

where matrixes E_α and X_β are defined by the relations (18) and (21) respectively. Then it follows easily that the norm

$$n(X) = c_{ijkp}.$$

Since the group F_4 can be isomorphically enclosed into the group E_6, we prove the E_6-invariance of the found solutions.

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